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# Polynomial Chaos Expansion with Latin Hypercube Sampling for Estimating Response Variability

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**A computationally efficient procedure for quantifying uncertainty and finding significant parameters of uncertainty models is presented. To deal with the random nature of input parameters of structural models, several efficient probabilistic methods are investigated. Specifically, the polynomial chaos expansion with Latin hypercube sampling is used to represent the response of an uncertain system. Latin hypercube sampling is employed for evaluating the generalized Fourier coefficients of the polynomial chaos expansion. Because the key challenge in uncertainty analysis is to find the most significant components that drive response variability, analysis of variance is employed to find the significant parameters of the approximation model. Several analytical examples and a large finite element model of a joined-wing are used to verify the effectiveness of this procedure.**

## Introduction

As modern structures require more critical and competitive designs, the need for accurate approaches to assess uncertainties in loads, geometry, material properties, manufacturing processes, and operational environments has increased significantly. A number of probabilistic analysis tools have been developed to quantify uncertainties, but the most complex systems are still designed with simplified rules and schemes, such as safety factor design. However, these traditional design processes do not directly account for the random nature of most input parameters. The purpose of the proposed approach is to better define and reduce uncertainties of systems by introducing a series of polynomials aimed to characterize the stochastic system being investigated. It will exploit several probabilistic design methodologies that have great potential in efficiency and accuracy. Although deterministic functions are applicable to mathematical models, the direct use of a random function expansion is useful for uncertainty analysis. The direct use of a random function, which is based on the concept of random process, provides analytically appealing convergence properties.[1,2] Therefore, polynomial chaos expansion (PCE) is used to represent the structural response. The PCE employs orthogonal polynomials of random variables; most commonly, the random variables are standard normal and the Hermite polynomials are used. The PCE is convergent in the mean-square sense, and any order PCE consists of all orthogonal polynomials.[2] This property can simplify the calculation of moments in statistical procedures. The Latin hypercube sampling (LHS) method[3] along with PCE supports the uncertainty analysis of large models in the current study. The following section describes several important properties of the PCE, LHS, and regression procedures.

Since the introduction of spectral stochastic finite element method by Ghanem and Spanos,[2] PCE has been successfully used to represent uncertainty in a variety of applications. Tatang[4] introduced the probabilistic collocation method in which the response of stochastic systems is projected onto the PCE with delta functions at each collocation point serving as the test functions in a Galerkin method. Coefficients of the PCE were obtained by using the model outputs at selected collocation points. Isukapalli[5] pointed out the limitation of the probabilistic collocation method for large-scale models and suggested a stochastic response surface method by using the partial derivatives of model outputs with respect to model inputs. To obtain the partial derivative of model outputs, ADIFOR, a FORTRAN programming library, was used in the stochastic response surface method. Recently, a nonintrusive formulation procedure was applied to the buckling eigenproblem by evaluating coefficients of the PCE through the Monte Carlo simulation (MCS).[6]

Each of these methods has some limitations. In the case of the probabilistic collocation method, especially for many PCE degrees of freedom, the available collocation points increases exponentially.[5] Therefore, many collocation points are not sampled. Consequently, the selected collocation points to obtain unknown coefficients of the PCE do not guarantee a space filling design (Fig. 1a), which fills up the available design space with specified sampling points according to the suitably defined design criteria, such as maximize minimum distance between points.[7] If we are interested in the tail regions of probability density function (Fig. 2), the data selection procedure should be reconsidered when applying the probabilistic collocation method because the selected design points are concentrated in the high probability region.

In the nonintrusive formulation procedure,[6] MCS is used to calculate the unknown coefficients of the PCE, but there is no a priori convergence criterion. Typically, the standard MCS requires a large number of simulations for an accurate estimate of model outputs. Thus, a desirable approach would be to combine the dependability of MCS with the efficiency of response surfaces modeling and to include verifiable convergence criteria.

In this paper, a new procedure is developed by using the analysis of variance (ANOVA)[8,9] and the LHS method,[3] which can guarantee each of the input variables has all portions of its range represented

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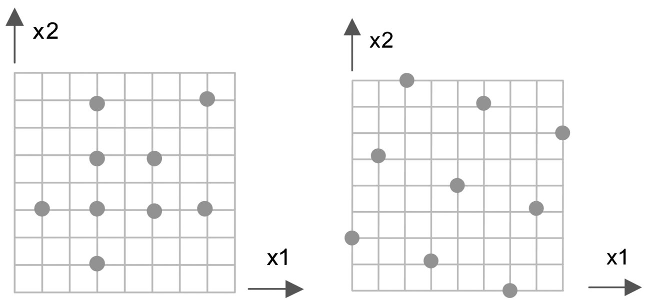



**a) Collocation points** **b) Restricted design of LHS**

**Fig. 1 Comparison of design points of LHS and probabilistic collocation method.**

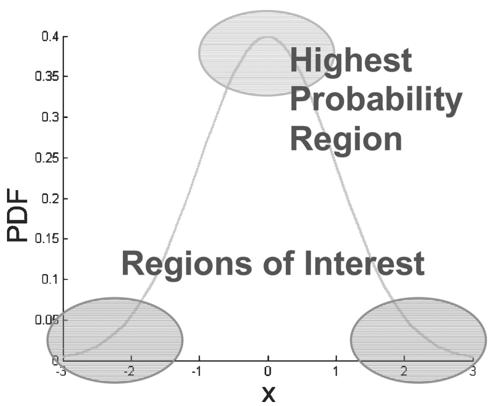



**Fig. 2 Regions of interest in PDF.**

(Fig. 1b). Novak et al.[10] showed the excellent superiority of LHS for complex computationally intensive problems as compared to MCS. Therefore, we can expect that the use of LHS can lead to a substantial reduction of numerical efforts in the current procedure. After an approximate model for uncertainty in a system is constructed, the significance test and the residual analysis are performed to check whether a sufficient fit has been achieved. The present procedure has specific steps for checking convergence criteria, which were not available to Pettit et al.[6] in their nonintrusive procedure. The current technique was successfully applied for non-Gaussian stochastic problems and compared to traditional methods including MCS and first-order reliability method (FORM). The result of PCE with LHS (100 simulations) produced a highly accurate agreement with MCS (10,000 simulations), whereas FORM failed to obtain sufficient accuracy.[11]

## Framework

The underlying concept for the hybrid procedure of solving the PCE coefficients is to build approximations for the response model as a PCE of the uncertain parameters. This procedure comprises the following steps:

1) Select experimental designs using LHS.
2) Simulate system response at each design point.
3) Construct approximate model using PCE.
4) Conduct ANOVA and residual analysis.

Once the approximation is constructed, the statistical properties of the response can be estimated. Then a general procedure of regression analysis, including testing for significance of regression, can be performed to find the significant components of the approximation model. The following sections provide details for each of these steps.

### PCE

The methodology is presented using a simple example. If we fit some data that have curvilinear responses, the polynomial regression model can be used:

$$Y(x) = \beta_0 F_0(x) + \beta_1 F_1(x) + \beta_2 F_2(x) + \beta_3 F_3(x) \tag{1}$$

In this model, $\beta_0$, $\beta_1$, $\beta_2$, and $\beta_3$ are the mean, linear, quadratic, and cubic effect of the responses, respectively. In general, the common polynomial model, which combines Eqs. (1) and (2), is useful for approximations of unknown or complex nonlinear relationships:

$$F_0(x) = 1, \qquad F_1(x) = x, \qquad F_2(x) = x^2, \qquad F_3(x) = x^3 \tag{2}$$

However, this often is not the best choice because these functions are not orthogonal. For instance, large positive values of $x$ give large positive values of all of the functions. If $x$ is negative, all of the odd power terms yield negative values and even power terms give positive values. Because they are not orthogonal, small changes of $F(x)$ would produce relatively large changes in the coefficients $\beta_0, \ldots, \beta_3$. This collinearity will make the associated least-squares problem ill conditioned. A more satisfactory solution is to use orthogonal polynomials, that is, Chebyshev polynomials, in this case,

$$F_0(x) = 1, \qquad F_1(x) = x$$
$$F_2(x) = 2x^2 - 1, \qquad F_3(x) = 4x^3 - 3x \tag{3}$$

**Table 1 Orthogonal polynomials[12,13]**

| PDF | Orthogonal polynomial | Support range |
|---|---|---|
| | *Continuous* | |
| Gaussian | Hermite | $(-\infty, +\infty)$ |
| Gamma | Laguerre | $[0, +\infty)$ |
| Beta | Jacobi | $[a, b]$ |
| Uniform | Legendre | $[a, b]$ |
| | *Discrete* | |
| Poisson | Charlier | $\{0, 1, 2, \ldots\}$ |
| Binomial | Krawtchouk | $\{0, 1, 2, \ldots, N\}$ |
| Negative binomial | Meixner | $\{0, 1, 2, \ldots\}$ |
| Hypergeometric | Hahn | $\{0, 1, 2, \ldots, N\}$ |

Obviously, the use of orthogonal polynomials can eliminate high levels of collinearity and ill-conditioned problems. There are several orthogonal polynomials described in the literature whose orthogonality weighting functions match standard probability density functions (PDF). Table 1 shows PDF with their related orthogonal polynomials and ranges.[12,13] Hermite polynomials are uncorrelated when $x$ is Gaussian on $(-\infty, +\infty)$ and Laguerre polynomials are uncorrelated when $x$ is gamma distributed on $(0, +\infty)$. The basic idea of the current study is to select an appropriate basis function to represent the response of an uncertain system. The PCE, which employs orthogonal basis functions and is mean-square convergent, is a good choice for estimating the response variability of uncertain systems.

Since Wiener[14] introduced the concept of the homogeneous chaos, PCE has been used for the uncertainty analysis in several applications, for example, in Refs. 2, 15, and 16. Ghanem and Spanos[2] introduced a simple definition of the PCE as a convergent series:

$$u(\theta) = a_0\Gamma_0 + \sum_{i=1}^{\infty} a_{i1}\Gamma_1(\xi_{i1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2[\xi_{i1}(\theta), \xi_{i2}(\theta)]$$
$$+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3[\xi_{i1}(\theta), \xi_{i2}(\theta), \xi_{i3}(\theta)] + \cdots \tag{4}$$

where $\{\xi_i(\theta)\}_{i=1}^{\infty}$ is a set of independent standard Gaussian random variables, $\Gamma_P(\xi_{i1}, \ldots, \xi_{iP})$ is a set of multidimensional Hermite polynomials, usually called the polynomial chaos of order $p$, $a_{i1}, \ldots, a_{i_P}$ are deterministic constants, and $\theta$ is the random character of the quantities involved.

PCE is convergent in the mean-square sense[1] and the $p$th-order PCE consists of all orthogonal polynomials of order $p$, including any combination of $\{\xi_i(\theta)\}_{i=1}^{\infty}$; furthermore, $\Gamma_P \perp \Gamma_q$ for $p \neq q$. This orthogonality greatly simplifies the procedure of statistical calculations, such as moments. Therefore, PCE can be used to approximate non-Gaussian distributions using a least-squares scheme, for example, to compare the skewness and kurtosis of distributions.

The general expression to obtain the multidimensional Hermite polynomials is given by

$$\Gamma_P(\xi_{i1}, \ldots, \xi_{iP}) = (-1)^n \frac{\partial^n e^{-\frac{1}{2}\xi^T \xi}}{\partial \xi_{i1}, \ldots, \partial \xi_{iP}} e^{\frac{1}{2}\xi^T \xi} \tag{5}$$

where the vector $\boldsymbol{\xi}$ consists of $n$ Gaussian random variables $(\xi_{i1}, \ldots, \xi_{in})$.

Equation (4) can be written more simply as

$$u(\theta) = \sum_{i=0}^{P} b_i \Psi_i[\boldsymbol{\xi}(\theta)] \tag{6}$$

where $b_i$ and $\Psi_i[\boldsymbol{\xi}(\theta)]$ are identical to $a_{i1}, \ldots, a_{iP}$ and $\Gamma_P(\xi_{i1}, \ldots, \xi_{iP})$, respectively.

If the solution is known, the generalized Fourier coefficients $b_i$ can be evaluated from

$$b_i = \frac{\langle u(\theta)\Psi_i[\boldsymbol{\xi}(\theta)]\rangle}{\langle \Psi_i[\boldsymbol{\xi}(\theta)]\Psi_i[\boldsymbol{\xi}(\theta)]\rangle} \tag{7}$$

where $\langle\cdot\rangle$ indicates the expected value operation. This approach was applied in Ref. 6 in the nonintrusive formulation procedure by using MCS to evaluate the expected values. This equation is equivalent to the coefficient calculation procedure of general regression analysis described later in the Fitting Regression Model subsection; thus, the method proposed here provides an alternative to crude MCS for estimating the expansion coefficients.

In the one-dimensional case, we can expand the random response $u$ using orthogonal polynomials in $\xi$, which has a known probability distribution such as unit normal, $N[0, 1]$. If $u$ is a function of random variable $x$ that has the known mean $\mu_x$ and variance $\sigma_x^2$, then $\xi$ is a normalized variable:

$$\xi = (x - \mu_x)/\sigma_x \tag{8}$$

The expansion can be written as

$$u = b_0 + b_1\xi + b_2(\xi^2 - 1) + b_3(\xi^3 - 3\xi) + \cdots = \sum_{i=0}^{P} b_i \Psi_i \tag{9}$$

where $b_i$, $i = 0, 1, 2, \ldots, n$, are unknown coefficients and the set$\{\Psi_i\}$ contains the Hermite polynomials in the random variable $\xi$.

Generally, the one-dimensional Hermite polynomials are defined by

$$\Psi_n(\xi) = (-1)^n[\varphi^{(n)}(\xi)/\varphi(\xi)] \tag{10}$$

where $\varphi^{(n)}(\xi)$ is the $n$th derivative of the normal density, $\varphi(\xi) = 1/\sqrt{(2\pi)}\exp(-\xi^2/2)$. This is simply the single-variable version of Eq. (5). From Eq. (10), we can readily find

$$\{\Psi_i\} = \{1, \xi, \xi^2 - 1, \xi^3 - 3\xi, \xi^4 - 6\xi^2 + 3, \xi^5 - 10\xi^3 + 15\xi, \ldots\} \tag{11}$$

The orthogonal polynomials and $\xi$ satisfy

$$\Psi_0 = 1, \qquad \langle\Psi_i\rangle = 0, \qquad \langle\Psi_i\Psi_j\rangle = \langle\Psi_i^2\rangle\delta_{ij}, \qquad \forall \quad i, j$$
$$\langle\xi^0\rangle = 1, \qquad \langle\xi^k\rangle = 0 \quad \forall \quad k \text{ odd} \qquad \langle\xi^k\rangle = (k-1)\langle\xi^{k-2}\rangle \tag{12}$$

where $\langle\cdot\rangle$ indicates the expected value operation.

**LHS**

To improve the computational efficiency and the global accuracy of approximation, we introduced an efficient sampling scheme in place of MCS. The method of LHS, which has been successfully used to generate multivariate samples of statistical distributions, was first proposed by McKay et al.[17] LHS ensures that each of the input variables has all portions of its range represented while being computationally cheaper to generate and having relatively small variance in the response.[18]

Here are the brief steps for the general LHS method:

1) Divide the range of each variable into $n$ nonoverlapping intervals on the basis of equal probability.

2) Randomly select one value from each interval with respect to its probability density.

3) Repeat steps 1 and 2 until we have selected the value of all random variables such as $x_1, x_2, \ldots, x_k$.

4) Pair the $n$ values obtained for each $x_i$ with the $n$ values obtained for the other $x_{j\neq i}$ at random.

The LHS method provides flexible sample sizes while ensuring stratified sampling, that is, each of the input variables is sampled at $n$ levels. A more detailed description of LHS may be found in Ref. 3.

**Fitting Regression Models**

Regression analysis is the investigation of the functional relationship between two or more variables. Some specific aspects related to our framework of analysis are presented in this section. More complete details are available in Ref. 8.

Consider the linear regression model

$$y(x) = \beta_0 + \beta_1 f_1(x) + \cdots + \beta_k f_k(x) + \varepsilon \tag{13}$$

where $\beta_i$, $i = 0, 1, 2, \ldots, k$, are the regression coefficients and $\varepsilon$, the error of the model equation, is assumed to be normally distributed with mean zero and variance $\sigma_e^2$.

Equation (13) can be written in matrix notation for $n$ sample values of $x$ and $y$ as

$$Y = X\hat{\beta} + e \tag{14}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & f_1(x_1) & f_2(x_1) & \ldots & f_k(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \ldots & f_k(x_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & f_1(x_n) & f_2(x_n) & \ldots & f_k(x_n) \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}, \qquad e = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Generally, the method of least squares is used to obtain the regression coefficients:

$$\hat{\beta} = (X^T X)^{-1} X^T Y \tag{15}$$

The fitted model and the residuals are

$$\hat{Y} = X\hat{\beta}, \qquad e = Y - \hat{Y} \tag{16}$$

The covariance matrix of $\hat{\beta}$ is

$$\text{cov}(\hat{\beta}) = E\{(\hat{\beta} - E[\hat{\beta}])(\hat{\beta} - E[\hat{\beta}])^T\} = \sigma_e^2 (X^T X)^{-1} \tag{17}$$

where $E(\cdot)$ is the expected value or mean value.

The total sum of squares ($SS_t$) regression sum of squares ($SS_r$), and error (residual) sum of squares ($SS_e$) are given as

$$SS_t = Y^T Y \tag{18}$$

$$SS_r = \hat{Y}^T \hat{Y} = \hat{\beta}^T X^T Y \tag{19}$$

$$SS_e = e^T e \qquad \text{or} \qquad SS_e = SS_t - SS_r \tag{20}$$

The test of significance of regression involves ANOVA. This can determine the significant contributors of the model and can estimate the lack of fit and confidence interval on the mean response. The test procedure is usually summarized in an ANOVA table such as Table 2. The test statistic $F_0$ in Table 2 contributes to the significance test of the regression model. If the observed value of $F_0$ is larger than $F$ statistic, $F_0 > F_{\alpha,1,df_e}$, then the coefficient is judged to have a significant effect on the regression model. The $F$ statistic has two parameters in this case, denoted by $\alpha$ and $df_e$. The $df_e$ is the degrees of freedom of residual and $\alpha$ indicates the $100(1-\alpha)$th percentile of the $F$ distribution. The percentage points of the $F$ distribution for the specific degrees of freedom can be calculated and tabulated.[8] The plots of the residuals $e$ vs the corresponding fitted values $\hat{Y}$ or the observed values $Y$ vs $\hat{Y}$ are good measures for determining model adequacy. These graphical plots and other statistical tests, for example, normal probability plot,[8] yield the residual analysis, which can detect model inadequacies with little additional effort. The visual inspections of residuals are preferable to understand certain characteristics of the regression results because analysts can easily construct the plots and reveal useful information from the unorganized data. Example patterns of residual plots such as satisfactory, funnel, double bow, and nonlinear cases are available in Refs. 8 and 9. Abnormality of the residual plots would indicate that the selected model is inadequate or that there exists an error in analysis. When the residual analysis detects these common types of model inadequacies, the analysts require considerations of extra terms in the regression model, for example, higher-order or interaction terms.

**Table 2 ANOVA for significance of regression**

| Source of variance | Sum of squares | Degrees of freedom | Mean square (MS) | $F_0$ |
|---|---|---|---|---|
| Regression | $SS_r$ | $df_r = k$ | $MS_r = SS_r/df_r$ | $MS_r/MS_e$ |
| Residual | $SS_e$ | $df_e = n\text{-}k\text{-}1$ | $MS_e = SS_e/df_e$ | |
| Total | $SS_t$ | $df_t = n\text{-}1$ | | |

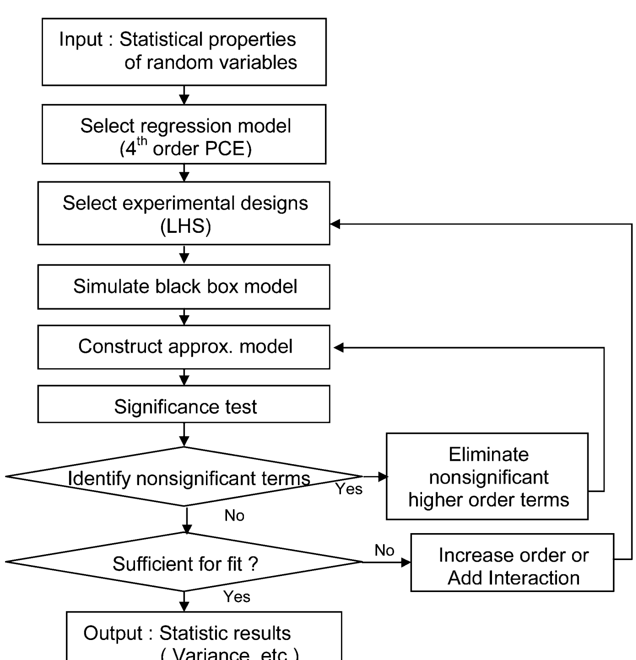



**Fig. 3 Backward elimination procedure.**

For selecting the appropriate order of approximation polynomials, we can proceed using either of the following two strategies.[8] One approach is a forward selection procedure, which involves increasing the polynomial order until the highest-order term is nonsignificant according to the significance test. The other approach is the backward elimination procedure, which is to fit a response model using the highest-order term and then delete terms one at a time. A detailed procedure of the later approach is shown in Fig. 3. To determine the significant and nonsignificant terms of the regression model, the $F$ statistics was used. Thus, the $\alpha$ value of $F$ statistics can indicate the acceptance and rejection levels of the regressors. Typically, the $\alpha$ of 0.05 and 0.10 are common choices for both the acceptance and rejection levels; however, these values can be changed according to the analyst's experience. Some researchers prefer to set the rejection level $\alpha$ to be larger than the acceptance level $\alpha$ to protect the rejection of regressors that are already admitted. Alternative methods, which involve $R^2$, $s^2$, and $C_p$ statistics, select the best regression equations, and further discussions of their use and advantages may be found in Ref. 9. The following sections clearly demonstrate the forward selection and backward elimination procedures.

## Application of Hybrid PCE Procedure

As described earlier, our framework consists of four parts: sampling design points, simulation, approximating the regression model, and ANOVA. We give two simple examples to demonstrate these four steps and then apply the concepts to a more complex model of a joined-wing structure.

### Demonstration Examples

To explain the hybrid procedure of PCE employed here, we will use a simple model that has one normal random variable $x$, which has a mean of 2.0 and unit standard deviation:

$$Y = e^x \tag{21}$$

Generally, a lower-order model is preferred to a higher-order model in regression analysis because an arbitrary fit of higher-order polynomials may create serious errors.[8] Therefore, choosing the order of an approximation is an important part of regression analysis. To identify this, we use the significance test and residual analysis. In this example, we will use the forward selection procedure to fit models of increasing order until the significance test for the highest-order term is nonsignificant. We can initially set the approximate model of $Y$ by introducing the second-order polynomials

$$\hat{Y} = \beta_0 F_0(\xi) + \beta_1 F_1(\xi) + \beta_2 F_2(\xi) \tag{22}$$

where $\xi$ has a standard normal distribution, $N(0, 1)$.

For an efficient approach, we choose orthogonal polynomials such as the PCE:

$$F_0(\xi) = 1, \qquad F_1(\xi) = \xi, \qquad F_2(\xi) = \xi^2 - 1 \tag{23}$$

The random variable $x$ is transformed as

$$x = \mu_x + \sigma_x F_1(\xi) \tag{24}$$

Thus, from the given distribution parameters, $x$ can be written as

$$x = 2 + F_1(\xi) \tag{25}$$

To find the unknown coefficients $\beta_0$–$\beta_2$ of the approximate model, LHS is used to identify the input design points. The number of input points must be higher than the number of unknown coefficients; generally, about twice the number of input points is sufficient for a simple case like this. For example, suppose that the five input points $\{x_1, x_2, \ldots, x_5\}$ are {1.3598, 4.0560, −0.3177, 2.5972, 1.9640}, with corresponding $\xi = \{-0.6402, 2.0560, -2.3177, 0.5972, -0.0360\}$. The corresponding responses are $Y = \{3.8954, 57.7439, 0.7278, 13.4263, 7.1274\}$. The unknown

**Table 3 One-dimensional example ANOVA (second-order PCE)**

| Source of variance | SS | Degrees of freedom | MS | $F_0$ |
|---|---|---|---|---|
| Regression | 3492.90 | 3 | 1164.30 | 26.40 |
| $\beta_0 F_0(\xi)$ | 750.78 | 1 | 750.78 | 17.02 |
| $\beta_1 F_1(\xi)$ | 1986.84 | 1 | 1986.84 | 45.04 |
| $\beta_2 F_2(\xi)$ | 755.29 | 1 | 755.29 | 17.12 |
| Residual | 88.22 | 2 | 44.11 | |
| Total | 3581.12 | 5 | | |

**Table 4 One-dimensional example ANOVA (third-order PCE)**

| Source of variance | SS | Degrees of freedom | MS | $F_0$ |
|---|---|---|---|---|
| Regression | 4291.16 | 4 | 1072.79 | 8.60 |
| $\beta_0 F_0(\xi)$ | 1070.53 | 1 | 1070.53 | 8.58 |
| $\beta_1 F_1(\xi)$ | 1773.05 | 1 | 1773.05 | 14.22 |
| $\beta_2 F_2(\xi)$ | 1277.46 | 1 | 1277.46 | 10.24 |
| $\beta_3 F_3(\xi)$ | 170.12 | 1 | 170.12 | 1.36 |
| Residual | 374.14 | 3 | 124.71 | |
| Total | 4665.30 | 7 | | |

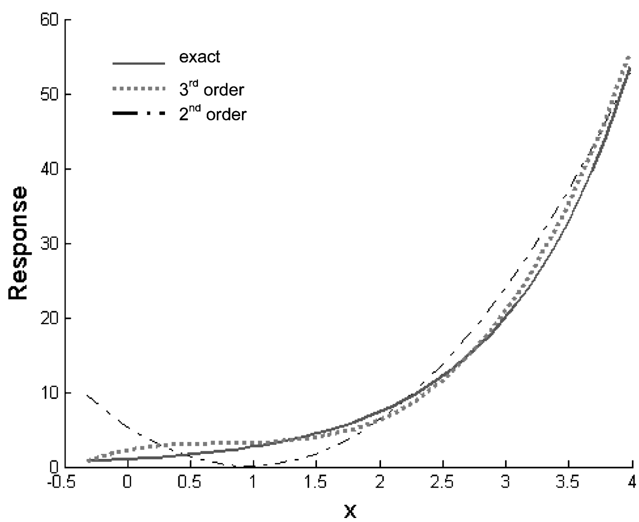



**Fig. 4 Approximate vs exact response for one-dimensional example.**

coefficients $\beta_0$–$\beta_2$ found using regression analysis give the following regression function:

$$\hat{Y} = 12.2538 + 13.8439 \times F_1(\xi) + 4.9133 \times F_2(\xi) \tag{26}$$

The exact and approximate responses are shown in Fig. 4. The second-order approximation is very poor in the interval [0, 2]. The cubic polynomials produce more accurate results over the entire interval. The ANOVA is summarized in Table 3. We did not use the residual plots because the visual comparison of the response is quite easy and more intuitive for this simple case. ANOVA shows that all $\beta$ terms are significant when we select an $F_{0.10,1,2}$ value of 8.53, which is smaller than the $F_0$ values in Table 3. Therefore, we should check the effect of higher-order terms of our regression model.

Figure 4 also shows the approximation results of a third-order PCE model, and Table 4 illustrates the ANOVA of the third-order case with seven simulations using the same procedure as the second-order case. The regression coefficients $\beta_0$–$\beta_3$ of the third-order case are {12.3667, 11.8221, 5.6240, 1.5042}. If we choose the $F_{0.10,1,3}$ value of 5.54, the regression coefficients $\beta_0$–$\beta_2$ and the total regression are significant and $\beta_3$ is nonsignificant. Based on the $F$ value of the coefficient $\beta_3$, we can expect that the other higher terms should have trivial effects on the current model. Thus, we do not need to check the effect of higher-order terms in the regression. Because the contribution of the highest-order term is not significant, the third-order PCE is sufficient for fitting as shown in Fig. 4. In this example, we used the 90% ($\alpha = 0.10$) $F$ value for specific $df_e$ as a stopping rule to quit adding the polynomial orders in the forward selection procedure. According to the results of the $F$ statistics, we can identify a reasonable choice of polynomial orders in the regression model. If users want to use $F$ statistics as cutoff criteria, the analysts should keep mind that valuable regressors can be removed by the relatively large values of $F$. Choosing the cutoff values of $F$ statistics in stepwise regression procedure has been criticized in the literature and often depended on a matter of the personal preference of the analyst. Thus, inexperienced analysts may require more considerations of cutoff rules for stepwise procedure. Further discussions of this procedure and alternative algorithms are available in Refs. 8 and 9.

**Table 5 Two-dimensional example ANOVA (second-order PCE)**

| Source of variance | SS | Degrees of freedom | MS | $F_0$ |
|---|---|---|---|---|
| Regression | 987900.92 | 6 | 164650.15 | 43.13 |
| $\beta_0$ | 180947.67 | 1 | 180947.67 | 47.40 |
| $\beta_1 F_1(\xi_1)$ | 326465.17 | 1 | 326465.17 | 85.53 |
| $\beta_2 F_1(\xi_2)$ | 187782.45 | 1 | 187782.45 | 49.20 |
| $\beta_3 F_2(\xi_1)$ | 177967.54 | 1 | 177967.54 | 46.62 |
| $\beta_4 F_2(\xi_2)$ | 93736.60 | 1 | 93736.60 | 24.56 |
| $\beta_5 F_1(\xi_1) F_1(\xi_2)$ | 21001.49 | 1 | 21001.49 | 5.50 |
| Residual | 19085.47 | 5 | 3817.09 | |
| Total | 1006986.39 | 11 | | |

In this simple system, $Y = e^x$, we can check the acceptable error by plotting the response and comparing it to the Maclaurin polynomial. Because such visual comparisons are not available in practical problems, the procedure involving residual analysis will be applied to next examples. After finding the tolerable-order term of the regression model, we can proceed with the procedure of checking for sufficient fit using residual analysis. The following examples illustrate the procedure.

We will apply the same procedure to a two-dimensional case, $Y = e^{x_1 + x_2}$, where $x_1$ and $x_2$ are independent variables that are normally distributed with a mean of 2.0 and a standard deviation of 1.0. We use the approximate model of $Y$ by introducing polynomials that include the interaction of $x_1$ and $x_2$:

$$\hat{Y} = \beta_0 + \beta_1 F_1(\xi_1) + \beta_2 F_1(\xi_2) + \beta_3 F_2(\xi_1) + \beta_4 F_2(\xi_2) + \beta_5 F_1(\xi_1) F_1(\xi_2) \tag{27}$$

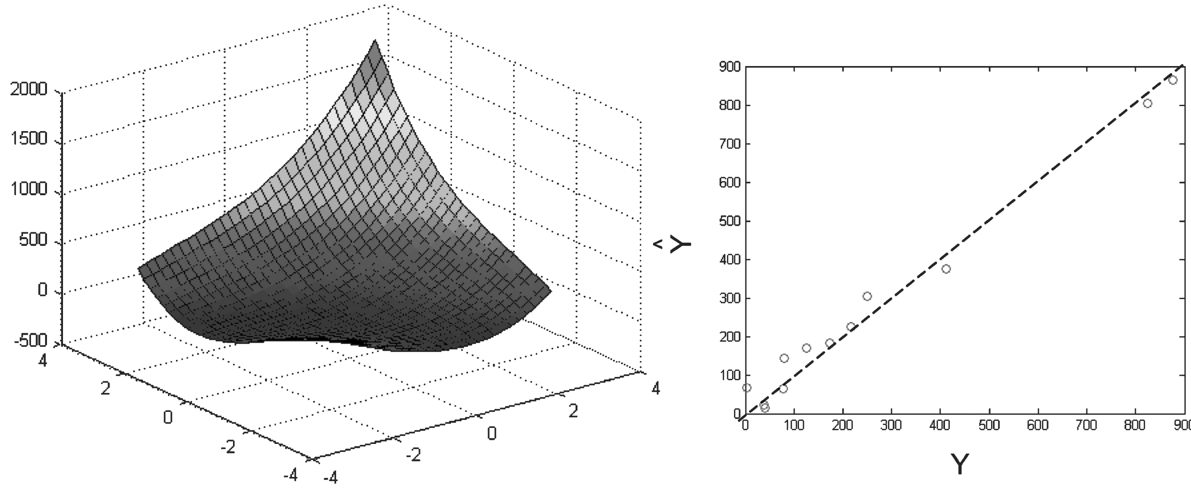
where $F_i(\xi)$, $i = 1, 2, \ldots, n$, are $n$th-order PCE.

Table 5 summarizes the ANOVA analysis of the two-dimensional case results. When we choose the $F_{0.10,1,5}$ value of 4.06 with respect to the given degree of freedom of the regression model, the $F_0$ column values are greater than $F$ value as shown in Table 5. Thus, we should consider the effect of higher-order terms of the given model. Figure 5 shows the results of third- and fourth-order cases. Though every coefficient is significant in the third-order case, the highest term is nonsignificant in the fourth-order case. We do not need to exploit the effects of higher-order terms based on the $F$ statistics. The plot of residuals vs $\hat{y}$, or $y$ vs $\hat{y}$, can provide a visual assessment of model effectiveness in regression analysis. Because both residual plots of Fig. 5 yield points close to the 45-deg line, the estimated regression function gives accurate predictions of the values that are actually observed. Therefore, the fourth-order PCE model is sufficient for fitting the given data, and we can obtain useful statistical properties using an approximation model.

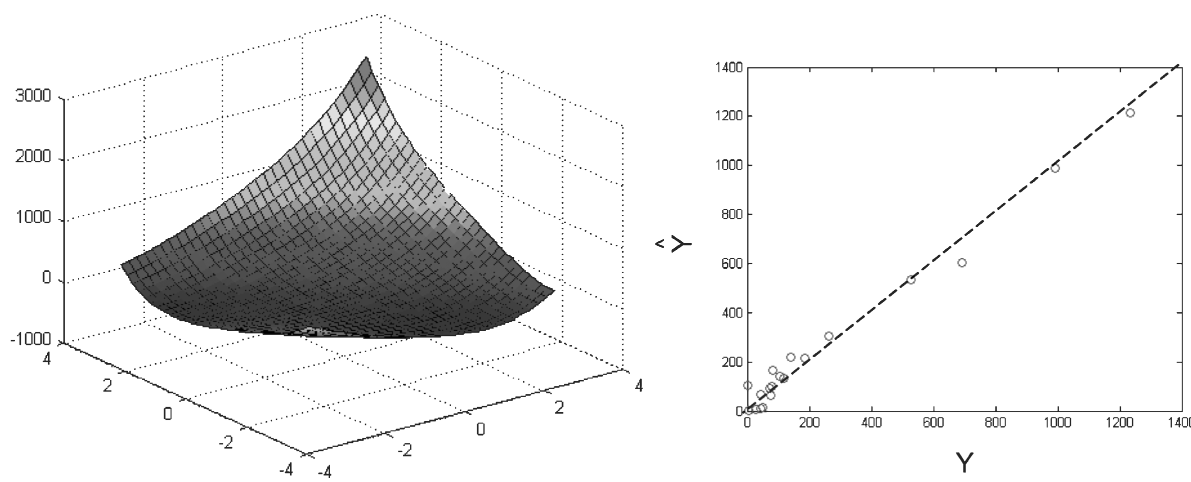
### Joined-Wing Example

To illustrate the practical use of the current procedure, we conduct a buckling analysis of the joined-wing model, which has been studied by Pettit et al.[6] In the design of the joined wing, the backward- and forward-swept lifting surfaces combine to replace the traditionally separate wing and horizontal tail. This nontraditional design potentially offers weight savings, reduces maneuver drag, and improves stability. The particular design issue of the joined wing is to overcome the buckling response of the aft wing and aeroelastic complications with the presence of compressive loads. A more detailed description of the joined-wing design can be found in Ref. 19.

The joined-wing structure model consists of 1562 grid points and 3013 elements in a geometrically nonlinear finite element model in NASTRAN, as shown in Fig. 6. In the current model, 2173 QUAD4,




a) Third-order PCE with 17 simulations




b) Fourth-order PCE with 25 simulations

**Fig. 5 Response and residual plots for two-dimensional example.**

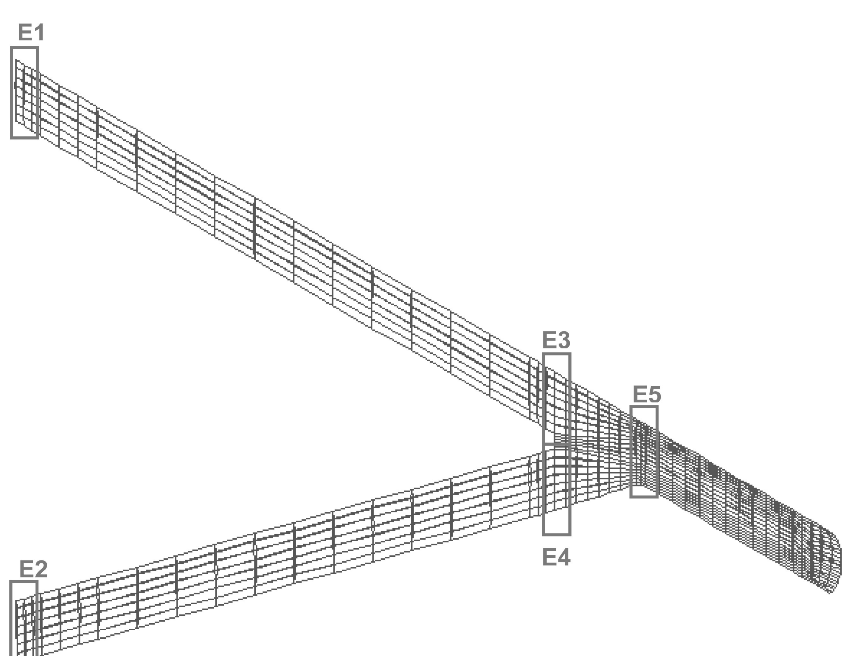



**Fig. 6 Joined-wing model and the locations of five random variable parts.**

156 TRIA3, and 684 SHEAR elements compose the wing skins, ribs, and the webs, respectively, in each wing section. The roots of the fore and aft wings have fully constrained boundary conditions, and all other grid points have rotational degrees of freedom. The distributed steady aerodynamic pressures are applied for input to the buckling analysis in the baseline model, which has 30 deg of leading-edge sweep, a semispan of 26.0 m, and a uniform chord length of 2.50 m. Blair and Canfield[19] describe the calculation of the aerodynamic pressures.

This unconventional configuration demands investigation of the coupling between buckling and aeroelastic instabilities to diagnose the variability of the response. Traditional design tools do little to capture the uncertain nature of the unconventional designs and their environment. We apply the current procedure to quantify variability of the joined-wing structural response, which is modeled by assuming stochastic in the wing joint and the wing roots because of their importance in determining the response characteristics of the coupled structure.

In the buckling analysis procedure, the buckling eigenvalue is the relevant output of the given joined-wing system. In the current model, the five locations were chosen to include uncertain material properties, which is limited to Young's modulus of the skin and spar and rib elements in the vicinity of the wing joint and the two wing roots. Figure 6 shows the five corresponding groups of elements: 1) forward wing root, 2) aft wing root, 3) forward wing joint, 4) aft wing joint, and 5) outboard wing joint. The associated Young's moduli are assumed to be Gaussian and uncorrelated. The Young's moduli, which have a coefficient of variation of 0.1 and a mean of $6.9 \times 10^{10}$ Pa, are denoted by $E_1$–$E_5$, respectively, as shown in Fig. 6.

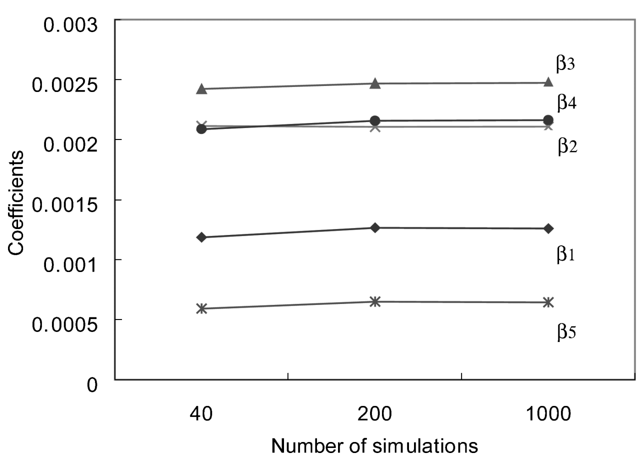



**Fig. 7 Coefficients convergence check with increased number of simulations.**

The eigenvalue problem can be expressed as a function of $\xi$, because Young's modulus is computed as $E = \mu_E + \sigma_E \xi$, that is,

$$[K(\xi) + \lambda(\xi)\Delta K(\xi)]\phi(\xi) = 0 \tag{28}$$

where $\Delta K$ represents the geometric stiffness matrix. The eigenvalues are projected onto the polynomial chaos expansion

$$\lambda = \sum_{i=0}^{P} \beta_i \Psi_i$$

then the current method or Eq. (7) can be applied to obtain the generalized Fourier coefficients. The associated moments of the eigenvalue can be obtained by the following expressions:

$$\langle \lambda \rangle = \beta_0, \qquad \mathrm{var}(\lambda) = \sum_{i=1}^{P} \langle \Psi_i^2 \rangle \beta_i^2 \tag{29}$$

where $\beta_i$ are the regression coefficients of polynomial chaos expansions $\Psi_i$, and $\langle \cdot \rangle$ indicates the expected value operation.

In the current model, we use the backward elimination procedure (Fig. 3), with a fourth-order PCE; therefore, we initially have 21 unknown coefficients in the regression model without considering the interaction effects. If the selected regression model does not satisfy model adequacy, we should consider the interaction terms or the other order terms as described in the preceding section and Fig. 3. To obtain the unknown regression coefficients of this model, we conducted 40 simulations based on LHS by using NASTRAN. According to the result of ANOVA, the higher-order terms (third and fourth) have no effect on our regression model because the values

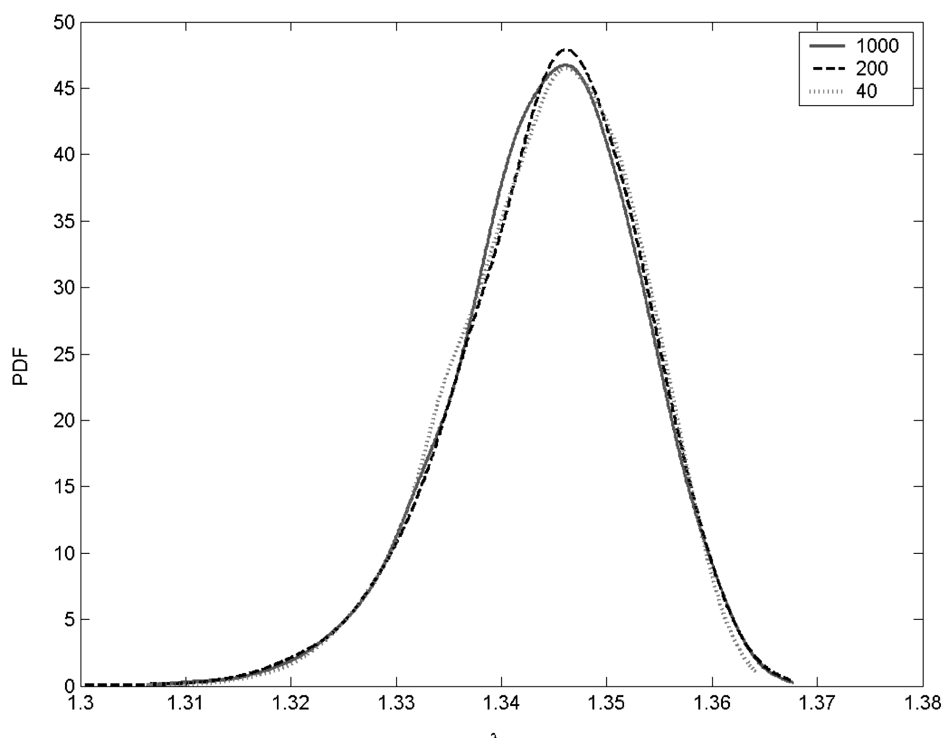



**Fig. 8 PDF of the first buckling eigenvalue as a function of the number of simulations.**

of these coefficients are almost zero ($-4.87 \times 10^{-5}$–$1.62 \times 10^{-5}$). Thus, we can eliminate these higher-order terms and use the selected model to obtain statistic properties because the current model, excluding the interaction terms, shows sufficient model adequacy. The current joined-wing model has first and second buckling eigenvalues of $\lambda_1 = 1.3445$ and $\lambda_2 = 1.8071$ with standard deviations of 0.00503 and 0.00519, respectively. The relatively small changes observed in the current model are natural because the buckling eigenvalue exhibits marked sensitivity only when applying large stiffness reductions.[6] The outer wing joint shows the lowest effect on the eigenvalues. The convergence of each coefficient was checked by comparing with coefficients from 200 and 1000 simulations, as shown in Fig. 7. After 200 simulations, all coefficients are converged with 0.1–0.5% of the 1000 simulation results. In the case of 40 simulations, the mean term $\beta_0$ has exactly the same value and the other coefficients have 0.33–8.28% difference compared with 1000 simulation results. Figure 8 shows the PDFs resulting from the MCS for the first buckling eigenvalue in each of the simulation results. All significant coefficients of the surrogate model are provided to obtain the PDF by running 10,000 simulations in MCS. Although the 5th coefficient of the 40 simulation results has the maximum error, Fig. 8 shows that this error does not severely alter the PDF.

## Conclusions

This paper has demonstrated an uncertainty analysis procedure and presented a study of a joined-wing aircraft that has uncorrelated random variables. Two case studies presented here demonstrate the procedure of the current method and illustrate the applicability to a large computational model. In the framework shown in Fig. 3, combining several modules relevant to probabilistic methods provides a computationally efficient procedure for uncertainty analysis. The proposed framework helps to identify the significant parameters of the uncertainty model and suggests a clear criterion for truncating the PCE. The approximation of stochastic response and the use statistical tests, for example, ANOVA and residual analysis, require intuition and experience with model behavior. Nevertheless, the method presented here is computationally efficient for structural uncertainty quantification. This procedure can improve analytical conclusions of more safety system and performance because it can give an estimation of how much a given input factor may drive the risk of the system according to the statistic results. In addition, there are no extra computational costs to obtain the statistical properties of the responses after constructing the approximation of stochastic responses. Therefore, the present approach can be a valuable tool in the uncertainty analysis because it is clear that the investigation of significant parameters of the stochastic model is critical and the deterministic tools do not reflect the stochastic nature of system behavior.

A future area of research will be to propagate other uncertain parameters of the joined-wing model and correlated random variables. To consider the correlation patterns of random variables, a restricted LHS,[3] which can induce any desired rank-correlation structure, or a decomposition technique,[10] which can decompose a target correlation structure using Karhunen–Loeve expansion, can be incorporated in this methodology.

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## References

[1]Cameron, R. H., and Martin, W. T., "The Orthogonal Development of Nonlinear Functionals in Series of Fourier–Hermite Functionals," *Annals of Mathematics*, Vol. 48, 1947, pp. 385–392.

[2]Ghanem, R., and Spanos, P. D., *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, 1991.

[3]Wyss, G. D., and Jorgensen, K. H., "A User's Guide to LHS: Sandia's Latin Hypercube Sampling Software," Sandia National Labs., Rept. SAND98-0210, Albuquerque, NM, Feb. 1998.

[4]Tatang, M. A., "Direct Incorporation of Uncertainty in Chemical and Environmental Engineering Systems," Ph.D. Dissertation, Chemical Engineering, Massachusetts Inst. of Technology, Cambridge, MA, Feb. 1995.

[5]Isukapalli, S. S., "Uncertainty Analysis of Transport-Transformation Models," Ph.D. Dissertation, Rutgers, Dept. of Chemical and Biomechanical Engineering, State Univ. of New Jersey, New Brunswick, NJ, Jan. 1999.

[6]Pettit, C. L., Canfield, R. A., and Ghanem, R., "Stochastic Analysis of an Aeroelastic System," 15th American Society of Civil Engineers Engineering Mechanics Conf., June 2002.

[7]Park, J. S., "Optimal Latin-Hypercube Designs for Computer Experiments," *Journal of Statistical Planning and Inference*, Vol. 39, No. 1, 1994, pp. 95–111.

[8]Montgomery, D. C., and Peck, E. A., *Introduction to Linear Regression Analysis*, Wiley, New York, 1992.

[9]Draper, N. R., and Smith, H., *Applied Regression Analysis*, Wiley, New York, 1981.

[10]Novak, D., Lawanisut, W., and Bucher, C., "Simulation of Random Fields Based on Orthogonal Transformations of Covariance Matrix and Latin Hypercube Sampling," *Proceedings of the International Conference on Monte Carlo Simulation*, Swets and Zeitlinger, Lisse, The Netherlands, 2000.

[11]Choi, S., Grandhi, R. V., and Canfield, R. A., "Propagation of Non-Gaussian Stochastic Behavior Using the Polynomial Chaos Expansion," *Second M.I.T. Conference on Computational Fluid and Solid Mechanics*, Massachusetts Inst. of Technology, Cambridge, MA, 2003, pp. 1896–1899.

[12]Xiu, D., and Karniadakis, G., "Modeling Uncertainty in Steady State Diffusion Problems via Generalized Polynomial Chaos," *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, 2002, pp. 4927–4948.

[13]Xiu, D., and Karniadakis, G., "The Wiener–Askey Polynomial Chaos for Stochastic Differential Equations," *SIAM Journal on Scientific Computing*, Vol. 24, No. 2, 2002, pp. 619–644.

[14]Wiener, N., "The Homogeneous Chaos," *American Journal of Mathematics*, Vol. 60, 1938, pp. 897–936.

[15]Imamura, T., and Meecham, W., "Wiener–Hermite Expansion in Model Turbulence in the Late Decay Stage," *Journal of Mathematical Physics*., Vol. 6, No. 5, 1965, pp. 707–721.

[16]Imamura, T., Meecham, W., and Siegel, A., "Symbolic Calculus of the Wiener Process and Wiener–Hermite Expansion," *Journal of Mathematical Physics*., Vol. 6, No. 5, 1965, pp. 695–706.

[17]McKay, M. D., Beckman, R. J., and Conover, W. J., "A Comparison of Three Methods for Selecting Values of Input Variables in the Analysis of Output from a Computer Code," *Technometrics*, Vol. 21, No. 2, 1979, pp. 239–245.

[18]Sacks, J., Schiller, S. B., and Welch, W. J., "Designs for Computer Experiments," *Technometrics*, Vol. 31, No. 1, 1989, pp. 41–47.

[19]Blair, M., and Canfield, R. A., "A Joined-Wing Structural Weight Modeling Study," AIAA Paper 2002-1337, April 2002.

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